algorithms and data structures (perfect matchings)

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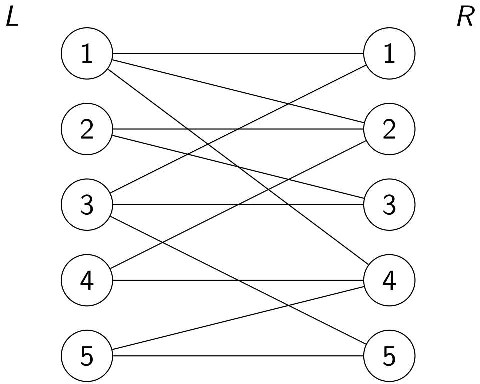
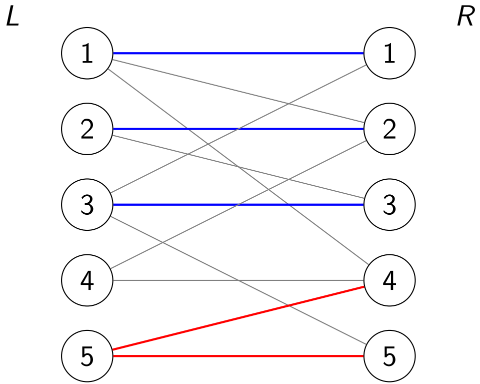
best way to match (e.g. workers–companies)

bipartite graph $G = (L, R, E)$



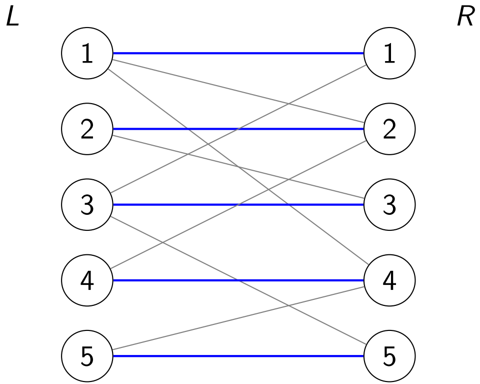
matchings

$M \subseteq E$ is a matching if $e \neq e' \in M \Rightarrow e \cap e' = \emptyset$

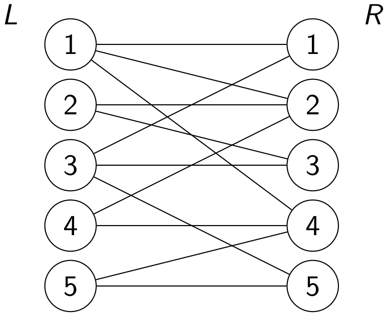


BPMs

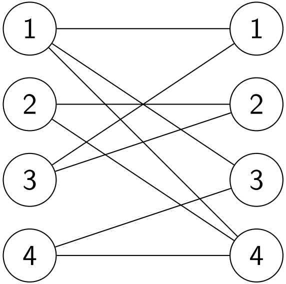
matching M is perfect if for every v there is $e \in M$ that contains v



input: bipartite graph $G = (L, R, E)$ with $|L| = |R| = n$
output: is there a PM?



adjacency matrix



$$A_G = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

input: matrix A

normalize rows:

for $i \in L$
  compute $r_i = \sum_{j \in R} A_{i,j}$

output the matrix $row(A)$ defined by

$$\frac{A_{i,j}}{r_i}$$

example

$$\begin{pmatrix} 2 & 0 & 1 & 0 & 3 \\ 0 & 4 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 3 & 2 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow[\textit{rows}]{} \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{2} \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{6} \\ \frac{4}{5} & 0 & 0 & \frac{1}{5} & 0 \end{pmatrix} \xrightarrow[\textit{columns}]{} \begin{pmatrix} \frac{20}{83} & 0 & \frac{1}{3} & 0 & \frac{3}{7} \\ 0 & \frac{8}{17} & 0 & \frac{5}{8} & 0 \\ \frac{15}{83} & \frac{3}{17} & 0 & 0 & \frac{3}{7} \\ 0 & \frac{6}{17} & \frac{2}{3} & 0 & \frac{1}{7} \\ \frac{48}{83} & 0 & 0 & \frac{3}{8} & 0 \end{pmatrix}$$

Linial–Samorodnitsky–Wigderson

input: matrix $A = A_G$

algorithm:

for n^5 times:[1]
    normalize rows
    normalize columns

if all row sums are close to one
    "accept"
else
    "reject"

[1] $O(n^2 \log n)$ also works

Linial–Samorodnitsky–Wigderson

input: matrix $A = A_G$

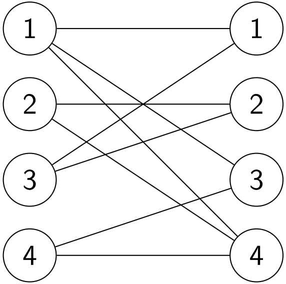
algorithm:

for n^5 times:
  normalize rows
  normalize columns

compute $r_i = \sum_{j \in R} A_{i,j}$

if $\sum_{i \in L}(r_i - 1) < \frac{1}{n}$
  output "accept"
else
  output "reject"

example



$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

the iterations

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \longrightarrow \begin{pmatrix} 0.40 & 0 & 0.40 & 0.25 \\ 0 & 0.50 & 0 & 0.38 \\ 0.60 & 0.50 & 0 & 0 \\ 0 & 0 & 0.60 & 0.38 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 0.38 & 0 & 0.38 & 0.24 \\ 0 & 0.57 & 0 & 0.43 \\ 0.55 & 0.45 & 0 & 0 \\ 0 & 0 & 0.62 & 0.38 \end{pmatrix} \longrightarrow \begin{pmatrix} 0.41 & 0 & 0.38 & 0.23 \\ 0 & 0.56 & 0 & 0.41 \\ 0.59 & 0.44 & 0 & 0 \\ 0 & 0 & 0.62 & 0.37 \end{pmatrix} \longrightarrow \begin{pmatrix} 0.40 & 0 & 0.37 & 0.22 \\ 0 & 0.58 & 0 & 0.42 \\ 0.57 & 0.43 & 0 & 0 \\ 0 & 0 & 0.63 & 0.37 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 0.41 & 0 & 0.37 & 0.22 \\ 0 & 0.57 & 0 & 0.42 \\ 0.59 & 0.43 & 0 & 0 \\ 0 & 0 & 0.63 & 0.37 \end{pmatrix} \longrightarrow \begin{pmatrix} 0.41 & 0 & 0.37 & 0.22 \\ 0 & 0.58 & 0 & 0.42 \\ 0.58 & 0.42 & 0 & 0 \\ 0 & 0 & 0.63 & 0.37 \end{pmatrix} \longrightarrow \begin{pmatrix} 0.42 & 0 & 0.37 & 0.22 \\ 0 & 0.58 & 0 & 0.42 \\ 0.58 & 0.42 & 0 & 0 \\ 0 & 0 & 0.63 & 0.37 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 0.41 & 0 & 0.37 & 0.22 \\ 0 & 0.58 & 0 & 0.42 \\ 0.58 & 0.42 & 0 & 0 \\ 0 & 0 & 0.63 & 0.37 \end{pmatrix} \longrightarrow \begin{pmatrix} 0.42 & 0 & 0.37 & 0.21 \\ 0 & 0.58 & 0 & 0.42 \\ 0.58 & 0.42 & 0 & 0 \\ 0 & 0 & 0.63 & 0.37 \end{pmatrix}$$

“accept”

matrix	row sums	column sums
	(3, 2, 2, 2)	(2, 2, 2, 3)
1		(0.83, 1.00, 0.83, 1.33)
2	(1.05, 0.88, 1.10, 0.98)	
3		(0.93, 1.02, 1.00, 1.05)
4	(1.02, 0.97, 1.03, 0.99)	
5		(0.97, 1.01, 1.00, 1.01)
6	(1.00, 0.99, 1.02, 1.00)	
7		(0.99, 1.00, 1.00, 1.01)
8	(1.01, 1.00, 1.00, 1.00)	
9		(0.99, 1.00, 1.00, 1.01)
10	(1.00, 1.00, 1.00, 1.00)	

observation: running time is poly(n)

theorem: the algorithm is correct

but why?

need to show

if "accept" then there is PM

if there is PM then "accept"

existence

denote by r_i and c_j the row and column sums of the final A

theorem

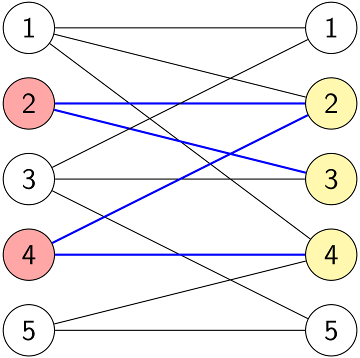
if $\sum_i |r_i - 1| < \frac{1}{n}$ and $\sum_j |c_j - 1| = 0$ then there is PM in G

in other words: if "accept" then there is PM

theorem

G has a PM iff $|N(S)| \geq |S|$ for all $S \subseteq L$

where $N(S) = \{j \in R : \exists i \in S \ (i, j) \in E\}$



Hall

theorem

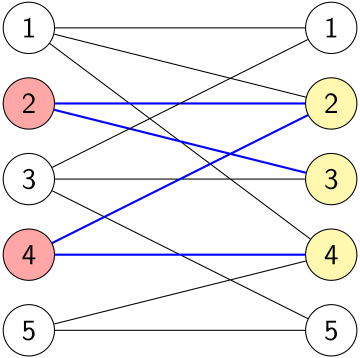
G has a PM iff $|N(S)| \geq |S|$ for all $S \subseteq L$

where $N(S) = \{j \in R : \exists i \in S \; (i, j) \in E\}$

proof

$\Rightarrow$ obvious

$\Leftarrow$ induction, max-flow min-cut, ...



if “accept” then PM

theorem if $\sum_i |r_i - 1| < \frac{1}{n}$ and $\sum_j |c_j - 1| = 0$ then PM

if “accept” then PM

theorem if $\sum_i |r_i - 1| < \frac{1}{n}$ and $\sum_j |c_j - 1| = 0$ then PM

proof: let $S \subseteq L$

if "accept" then PM

theorem if $\sum_i |r_i - 1| < \frac{1}{n}$ and $\sum_j |c_j - 1| = 0$ then PM

proof: let $S \subseteq L$

by assumption

$$\left| |S| - \sum_{i \in S} r_i \right| < 1 \quad \text{and} \quad |N(S)| = \sum_{j \in N(S)} c_j$$

if "accept" then PM

theorem if $\sum_i |r_i - 1| < \frac{1}{n}$ and $\sum_j |c_j - 1| = 0$ then PM

proof: let $S \subseteq L$

by assumption

$$\left| |S| - \sum_{i \in S} r_i \right| < 1 \quad \text{and} \quad |N(S)| = \sum_{j \in N(S)} c_j$$

so

$$|S| - 1 < \sum_{i \in S} \underbrace{\sum_{j \in N(S)} A_{i,j}}_{r_i} \leq \sum_{j \in N(S)} \underbrace{\sum_{i \in L} A_{i,j}}_{c_j} = |N(S)|$$

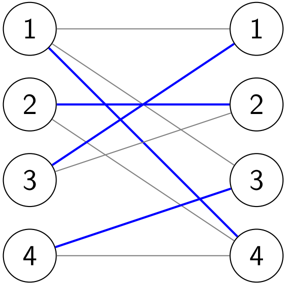
by Hall's theorem if "accept" then there is PM

main theorem

if there is PM then "accept"

observation: G has PM iff there is a bijection $\pi : L \to R$ such that

$$\prod_{i \in L} A_{i,\pi(i)} = 1$$



$$A = \begin{pmatrix} 1 & 0 & 1 & \mathbf{1} \\ 0 & \mathbf{1} & 0 & 1 \\ \mathbf{1} & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 1 \end{pmatrix}$$

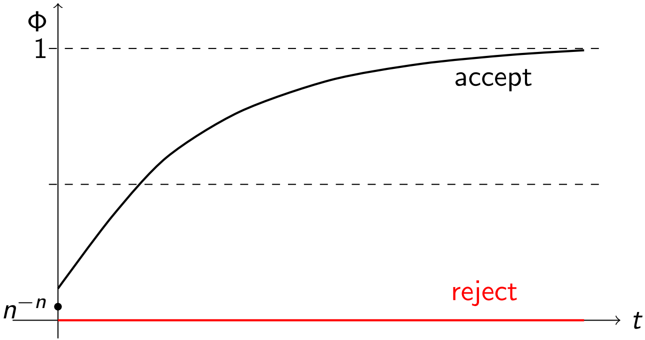
a potential

if G has PM that corresponds to $\pi : L \to R$ define

$$\Phi(A) := \prod_{i \in L} A_{i,\pi(i)}$$

comments
can think of π as identity map
exists only if there is PM

potential



$A_0 = col(A_G)$
A is column-normalized with row sums a_i
B is $col(row(A))$ with row sums b_i

lemma

1. if there is PM then $\Phi(A_0) \geq n^{-n}$
2. $\Phi(B) \geq \Phi(A)$
3. $\sum_i |b_i - 1| \leq \sum_i |a_i - 1|$
4. if $\sum_i |a_i - 1| = \delta$ then (with $\epsilon = \frac{\delta^2}{8n}$)
$$\Phi(B) \geq (1+\epsilon)\Phi(A)$$
5. $\Phi(B) \leq 1$

correctness

sequence of matrices $A_0, A_1, A_2, \ldots, A_T$

1. if there is PM then $\Phi(A_0) \geq n^{-n}$
2. Φ grows

3+4. if final test fails then $\Phi(A_{t+1}) \geq (1 + \frac{1}{8n^3})\Phi(A_t)$

5. $\Phi(A_T) \leq 1$

sequence of matrices $A_0, A_1, A_2, \dots, A_T$

1. if there is PM then $\Phi(A_0) \geq n^{-n}$
2. Φ grows

3+4. if final test fails then $\Phi(A_{t+1}) \geq (1 + \frac{1}{8n^3})\Phi(A_t)$

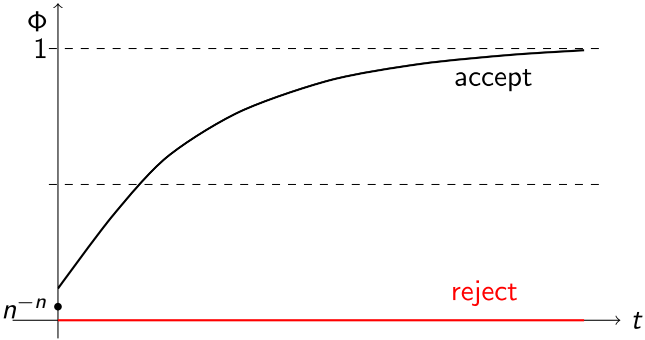
5. $\Phi(A_T) \leq 1$

if G has PM then

$$(1 + \tfrac{1}{8n^3})^T n^{-n} \leq 1$$

so $T \leq 8n^4 \log n$

potential



update

claim

if $B = row(A)$ then

$$\Phi(B) = \frac{1}{\prod_i r_i}\Phi(A)$$

reason

row i is divided by r_i

each row appears exactly once

initially

A_G is boolean so $c_j = c_j(A_G) \leq n$

$A_0 = col(A_G)$

$\Phi(A_G) = 1$

so

$$\Phi(A_0) = \frac{1}{\prod_j c_j}\Phi(A_G) \geq n^{-n}$$

A is column-normalized with row sums r_i and column sums c_j

claim (am-gm inequality)

$$\Big(\prod_i r_i\Big)^{1/n} \leq \frac{1}{n}\sum_i r_i = \frac{1}{n}\sum_j c_j = 1$$

potential grows

if A is column-normalized with row sums r_i and $B = row(A)$ then

$$\Phi(B) = \frac{1}{\prod_i r_i}\Phi(A) \geq \Phi(A)$$

because $\prod_i r_i \leq 1$

claim

A is column-normalized with row sums r_i
$B = row(A)$ with column sums c_j
then

$$\sum_j |c_j - 1| \leq \sum_i |r_i - 1|$$

proof

A is column-normalized with row sums r_i

$B = row(A)$ with column sums c_j

$$\begin{aligned}
\sum_j |c_j - 1| &= \sum_j \left| \left(\sum_i \frac{A_{i,j}}{r_i} \right) - 1 \right| \\
&= \sum_j \left| \sum_i \left(\frac{A_{i,j}}{r_i} - A_{i,j} \right) \right| \\
&= \sum_j \left| \sum_i A_{i,j} \left(\frac{1}{r_i} - 1 \right) \right| \\
&\leq \sum_j \sum_i A_{i,j} \frac{1}{r_i} |r_i - 1| \\
&= \sum_i |r_i - 1| \frac{1}{r_i} \sum_j A_{i,j} = \sum_i |r_i - 1|
\end{aligned}$$

claim (stability am-gm)

if $\frac{1}{n}\sum_i r_i = 1$ and

$$\left(\prod_i r_i\right)^{1/n} \geq 1 - \epsilon$$

then

$$\sum_i |r_i - 1| \leq 2\sqrt{n\epsilon}$$

in words: the only way averages are close is that numbers are close

will not prove

contra positive

if A is column-normal with row sums r_i and

$$\sum_i |r_i - 1| > 2\sqrt{n\epsilon}$$

then

$$\left(\prod_i r_i\right)^{1/n} < 1 - \epsilon$$

contra positive

if A is column-normal with row sums r_i and

$$\sum_i |r_i - 1| > 2\sqrt{n\epsilon}$$

then

$$\left(\prod_i r_i\right)^{1/n} < 1 - \epsilon$$

so $B = row(A)$

$$\Phi(B) = \frac{1}{\prod_i r_i}\Phi(A) > \frac{1}{1-\epsilon}\Phi(A)$$

below one

claim

if A is column-normalized then $\Phi(A) \leq 1$

reason

product of numbers that are at most one

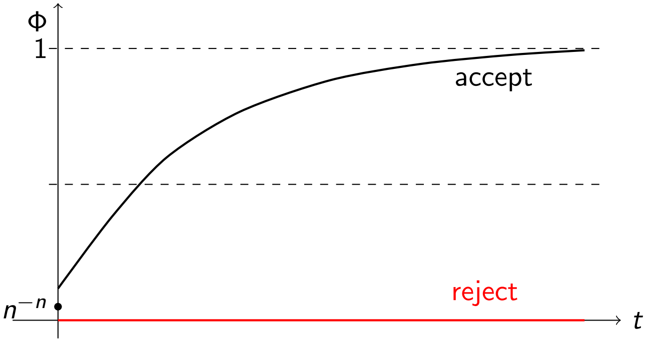
$$A_0 \to A_1 \to A_2 \to \cdots$$

1. if there is PM then $\Phi(A_0) \geq n^{-n}$
2. $\Phi(A_{t+1}) \geq \Phi(A)$
3. if $\sum_i |r_i(A_t) - 1| > \delta$ then

$$\Phi(A_{t+1}) \geq (1+\epsilon)\Phi(A_t)$$

4. $\Phi(A_t) \leq 1$

potential



starting at A_G the algorithm converges to a "limit" matrix A_*

what is A_*?

the limit

starting at A_G the algorithm converges to a "limit" matrix A_*

what is A_*?

it is a fractional perfect matching of maximum (Shannon) entropy starting at A_G the algorithm converges to a "limit" matrix A_*

what is A_*?

it is a fractional perfect matching of maximum (Shannon) entropy

comments
many combinatorial objects have fractional ("smooth") versions
objects of maximum entropy are often "special"

summary

simplest algorithm for bipartite perfect matching

correctness is deep

can be extended to more abstract scenarios

Sinkhorn scaling

input: $n \times n$ matrices with positive entries A and G such that $A_{i,j} = 0 \iff G_{i,j} = 0$

output: diagonal matrices D, D' such that

$$r_i(A') = r_i(G) \text{ and } c_j(A') = c_j(G)$$

where $A' = DAD'$

theorem D, D' exist

the space of matrices is X

the collection of (positive) diagonal matrices $\mathcal{D}$ is a group

for $A \in X$

$$row(A) = DA$$

where $D_{i,i} = \frac{1}{r_i(A)}$

orbits

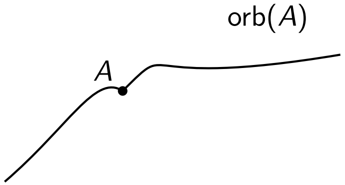
space X

group $\mathcal{D}$ acting on X

$D : X \to X$

$D(D'(A)) = (D \cdot D')(A)$

orbit of $A \in X$ is $orb(A) = \{D(A) : D \in \mathcal{D}\}$

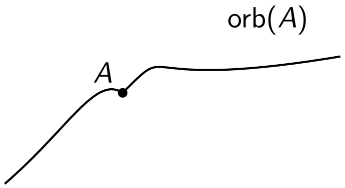


algorithmic question

space X

group $\mathcal{D}$ acting on X

task: given $A, B \in X$ is $B \in orb(A)$?



BPMs

space X of $n \times n$ matrices with positive entries

group $\mathcal{D} \times \mathcal{D}$ acting on X via

$$A \mapsto DAD'$$

we saw:

$$\text{graphs with PM} \iff \text{(closure) of the orbit of bistochastic matrices}$$